Introduction to max-algebra

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Abstract

We present an overview of max-algebra basics. In particular, we show how to find all eigenvalues and eigenvectors of a matrix. A brief account of advanced topics and open problems is included as well.

1 Definitions

We denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, $\overline{\overline{\mathbb{R}}} = \overline{\mathbb{R}} \cup \{+\infty\}$ and

$$a \oplus b = \max(a, b)$$

and

$$a \otimes b = a + b$$

for $a, b \in \overline{\overline{\mathbb{R}}}$. By definition

$$(-\infty) + (+\infty) = -\infty = (+\infty) + (-\infty)$$

This notation is of key importance in max-algebra since it enables us to formulate and in many cases also solve certain non-linear problems in a linear-like way. Max algebra has been studied by many authors from the 1960's and the reader is referred to [40], [2], [36] or [11] for more information about max-algebra, see also [23], [25], [28], [50], [53], [18], [38], [35], [5], [4].

In max-algebra the pair of operations $(\oplus, \otimes)$ is extended to matrices and vectors formally in the same way as in linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices with elements from $\overline{\overline{\mathbb{R}}}$ of compatible sizes, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j, $C = A \otimes B$ if $c_{ij} = \sum_{k}^{\oplus} a_{ik} \otimes b_{kj} = \max_k(a_{ik} + b_{kj})$ for all i, j and $\alpha \otimes A = A \otimes \alpha = (\alpha \otimes a_{ij})$ for $\alpha \in \overline{\overline{\mathbb{R}}}$.

We denote $-\infty$ by ε and for convenience we also denote by the same symbol any vector or matrix whose every component is ε. If $a \in \mathbb{R}$ then the symbol a^{-1} stands for $-a$.

So $2 \oplus 3 = 3$, $2 \otimes 3 = 5$, $4^{-1} = -4$,

$$(5, 9) \otimes \begin{pmatrix} -3 \\ \varepsilon \end{pmatrix} = 2$$

and the system

$$\begin{pmatrix} 1 & -3 \\ 5 & 2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

in conventional notation reads

$$\begin{aligned} \max(1 + x_1, -3 + x_2) &= 3 \\ \max(5 + x_1, 2 + x_2) &= 7 \end{aligned}$$

The possibility of working in a linear-like way is based on the fact that $(\overline{\mathbb{R}}, \oplus, \otimes)$ is a commutative idempotent semiring and $(\overline{\mathbb{R}}^n, \oplus)$ is a semimodule. Let us denote by I the square matrix, called the *unit matrix*, whose diagonal entries are 0 and off-diagonal ones are ε. Obviously, $A \otimes I = A = I \otimes A$ whenever I is of a suitable dimension.

We will assume that m and n are given integers, $m, n \geq 1$, and M and N will denote the sets $\{1, ..., m\}$ and $\{1, ..., n\}$, respectively. A square matrix is called *diagonal*, notation $diag(d_1, ..., d_n)$, if its diagonal entries are $d_1, ..., d_n \in \mathbb{R}$ and off-diagonal entries are ε. Thus $I = diag(0, ..., 0)$. Any matrix which can be obtained from the unit (diagonal) matrix by permuting the rows and/or columns will be called a *permutation matrix* (*generalised permutation matrix*). The position of generalised permutation matrices is slightly more special in max-algebra than in conventional linear algebra as they are the only matrices having an *inverse*:

Theorem 1.1 *[25] Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$. Then a matrix $B = (b_{ij})$ such that*

$$A \otimes B = I = B \otimes A \tag{1}$$

exists if and only if A is a generalized permutation matrix.

Clearly, if an inverse matrix to A exists then it is unique and we may therefore denote it by A^{-1}. If A is a square matrix then the iterated product $A \otimes A \otimes ... \otimes A$ in which the letter A stands k-times will be denoted as A^k. By definition $A^0 = I$ for any square matrix A.

The symbol a^k applies similarly to scalars, thus a^k is simply ka and $a^0 = 0$. This definition immediately extends to $a^x = xa$ for any real x.

The idempotency of $\oplus$ enables us to deduce (easily by induction) the following formula, specific for max-algebra:

Lemma 1.1 *The following holds for every $A \in \overline{\mathbb{R}}^{n \times n}$ and non-negative integer k :*

$$(I \oplus A)^k = I \oplus A \oplus A^2 \oplus ... \oplus A^k. \tag{2}$$

The columns (rows) of $A = (a_{ij}) \in \overline{\overline{\mathbb{R}}}^{m \times n}$ will be denoted by $A_1, ..., A_n$ $(a_1, ..., a_m)$. As an analogue to "stochastic", A will be called *column* (*row*) $\mathbb{R}$-*astic* [25] if $\sum_{i \in M}^{\oplus} a_{ij} \in \mathbb{R}$ for every $j \in N$ (if $\sum_{j \in N}^{\oplus} a_{ij} \in \mathbb{R}$ for every $i \in M$), that is when A has no ε column (no ε row). The matrix A will be called *doubly* $\mathbb{R}$-*astic* if it is both row and column $\mathbb{R}$-astic. Also, we will call a vector *finite* if none of its components is $-\infty$ or $+\infty$. Similarly for scalars.

2 About the ground set

The semiring $(\overline{\mathbb{R}}, \oplus, \otimes)$ could be introduced in more general terms as follows: Let $\mathcal{G}$ be an arbitrary linearly ordered commutative group (LOCG). Let us denote the group operation by $\otimes$ and the linear order by $\leq$. Thus $\mathcal{G} = (G, \otimes, \leq)$ where G is a set. We can then define $a \oplus b = \max(a, b)$ for $a, b \in G$ and set $\overline{G} = G \cup \{\varepsilon\}$ where ε is an adjoined element such that $\varepsilon < a$ for all $a \in G$. It is easily seen that $S = (\overline{G}, \oplus, \otimes)$ is an idempotent commutative semiring. Max-algebra as defined in Section 1.1 corresponds to the case when $\mathcal{G}$ is the additive group of reals, that is $\mathcal{G} = (\mathbb{R}, +, \leq)$ where $\leq$ is the natural ordering of real numbers. This group will be denoted by $\mathcal{G}_0$ and called the *principal interpretation* [25]. Let us consider a few other linearly ordered commutative groups (here $\mathbb{R}^+ [\mathbb{Q}^+, \mathbb{Z}^+]$ are the sets of positive reals (rationals, integers)):

$$
\begin{aligned}
\mathcal{G}_1 &= (\mathbb{R}, +, \geq) \\
\mathcal{G}_2 &= (\mathbb{R}^+, \cdot, \leq) \\
\mathcal{G}_3 &= (\mathbb{Z}, +, \leq) \\
\mathcal{G}_4 &= (\mathbb{Q}^+, \cdot, \leq) \\
\mathcal{G}_5 &= (\mathbb{Z}^+, +, \geq)
\end{aligned}
$$

Obviously both $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic with $\mathcal{G}_0$ (the isomorphism in the first case is $f(x) = -x$, in the second case it is $f(x) = \log(x)$). This paper presents results for max-algebra over the principal interpretation but due to the isomorphism these results immediately extend to max-algebra over $\mathcal{G}_1$ and $\mathcal{G}_2$. Many (but not all) of the results are applicable to general LOCG.

3 Matrices and digraphs

We will sometimes use the language of directed graphs (digraphs). A *digraph* is an ordered pair $D = (V, E)$ where V is a non-empty set (of *nodes*) and $E \subseteq V \times V$ (the set of *arcs*). A *subdigraph* of D is any digraph $D' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. If $e = (u, v) \in E$ for some $u, v \in V$ then we say that e is *leaving* u and *entering* v. Any arc of the form (u, u) is called a *loop*.

Let $D = (V, E)$ be a given digraph. A sequence $\pi = (v_1, ..., v_p)$ of nodes in D is called a *path* (in D) if $p = 1$ or $p > 1$ and $(v_i, v_{i+1}) \in E$ for all $i = 1, ..., p - 1$. The node v_1 is called the *starting node* and v_p the *endnode* of π, respectively. The number $p - 1$ is called the *length* of π and will be denoted by $l(\pi)$. If u is the starting node and v is the endnode of π then we say that π is a $u - v$ path. If there is a $u - v$ path in D then v is said to be *reachable* from u, notation $u \to v$. Thus $u \to u$ for any $u \in V$. A path $(v_1, ..., v_p)$ is called a *cycle* if $v_1 = v_p$ and $p > 1$ and it is called an *elementary cycle* if, moreover, $v_i \neq v_j$ for $i, j = 1, ..., p - 1, i \neq j$. If there is no cycle in D then D is called *acyclic*.

A digraph D is called *strongly connected* if $u \to v$ for all nodes u, v in D. A subdigraph D' of D is called a *strongly connected component* of D if it is a maximal strongly connected subdigraph of D. All strongly connected

components of a given digraph $D = (V, E)$ can be identified in $O(|V| + |E|)$ time [49]. Note that a digraph consisting of one node and no arc is strongly connected and acyclic, however if a strongly connected digraph has at least two nodes then it obviously cannot be acyclic. Because of this singularity we will have to assume in some statements that $n > 1$.

If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ then the symbols F_A (Z_A) will denote the digraphs with the node set N and arc sets $E = \{(i, j) ; a_{ij} > \varepsilon\}$ $(E = \{(i, j) ; a_{ij} = 0\})$. F_A (Z_A) will be called the *finiteness (zero) digraph* of A. If F_A is strongly connected then A is called *irreducible* and *reducible* otherwise.

Lemma 3.1 *If $A \in \overline{\mathbb{R}}^{n \times n}$ is irreducible and $n > 1$ then A is doubly $\mathbb{R}$-astic.*

Note that a matrix may be reducible even if it is doubly $\mathbb{R}$-astic (e.g. I).

Lemma 3.2 *If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is row or column $\mathbb{R}$-astic then F_A contains a cycle.*

A *weighted digraph* is $D = (V, E, w)$ where (V, E) is a digraph and $w : E \longrightarrow \mathbb{R}$. All definitions for digraphs are naturally extended to weighted digraphs. If $\pi = (v_1, ..., v_p)$ is a path in (V, E, w) then the *weight* of π is $w(\pi) = w(v_1, v_2) + w(v_2, v_3) + ... + w(v_{p-1}, v_p)$ if $p > 1$ and ε if $p = 1$. A cycle σ is called *positive* if $w(\sigma) > 0$. In contrast, σ is called a *zero cycle* if $w(v_k, v_{k+1}) = 0$ for all $k = 1, ..., p - 1$.

Given $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ the symbol D_A will denote the weighted digraph (N, E, w) where $F_A = (N, E)$ and $w(i, j) = a_{ij}$ for all $(i, j) \in E$. If $\pi = (i_1, ..., i_p)$ is a path in D_A then we denote $w(\pi, A) = w(\pi)$ and it now follows from the definitions that $w(\pi, A) = a_{i_1 i_2} + a_{i_2 i_3} + ... + a_{i_{p-1} i_p}$ if $p > 1$ and ε if $p = 1$.

4 The key players

The following problems play a central role in max-algebra:

Problem 4.1 (One-sided linear systems) *Given $A \in \overline{\mathbb{R}}^{m \times n}$ and $b \in \overline{\mathbb{R}}^m$, find all $x \in \overline{\mathbb{R}}^n$ satisfying*

$$A \otimes x = b.$$

Problem 4.2 (The eigenproblem) *Given $A \in \overline{\mathbb{R}}^{n \times n}$, find all $\lambda \in \overline{\mathbb{R}}$ (eigenvalues) and all $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ (eigenvectors) satisfying*

$$A \otimes x = \lambda \otimes x.$$

In this section we introduce the main tools for solving these problems: the *maximum cycle mean*, *transitive closures* and *conjugation*. The eigenproblem is discussed in Section 6 and the one-sided linear system problem is briefly discussed at the end of this section.

4.1 Maximum cycle mean

Everywhere in this paper, given $A \in \overline{\mathbb{R}}^{n \times n}$, the symbol $\lambda(A)$ will stand for the *maximum cycle mean* of A, that is if D_A has at least one cycle then

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A), \tag{3}$$

where the maximisation is taken over all cycles in D_A and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{l(\sigma)} \tag{4}$$

denotes the *mean* of a cycle σ. If D_A is acyclic we set $\lambda(A) = \varepsilon$. Thus $\lambda(A) > \varepsilon$ if A is irreducible and $n > 1$. A is called *definite* if $\lambda(A) = 0$ [16], [25].

Example 4.1 *If*

$$A = \begin{pmatrix} -2 & 1 & -3 \\ 3 & 0 & 3 \\ 5 & 2 & 1 \end{pmatrix}$$

then the cycle means of cycles of length 1 are $-2, 0, 1$, *of length 2 are* $2, 1, 5/2$, *of length 3 are* 3 *and* $2/3$. *Hence* $\lambda(A) = 3$.

The maximum cycle mean of a matrix is of fundamental importance in max-algebra because for any square matrix A it is the greatest (max-algebraic) eigenvalue of A and every eigenvalue of A is the maximum cycle mean of some principal submatrix of A. Note that if $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is row or column $\mathbb{R}$-astic then $\lambda(A) > \varepsilon$.

Lemma 4.1 $\lambda(A)$ *remains unchanged if the maximisation in (3) is taken over all elementary cycles.*

Lemma 4.2 *Let* $A \in \overline{\mathbb{R}}^{n \times n}$. *Then for every* $\alpha \in \mathbb{R}$ *the sets of arcs (and therefore also the sets of cycles) in* D_A *and* $D_{\alpha \otimes A}$ *are equal and* $\mu(\sigma, \alpha \otimes A) = \alpha \otimes \mu(\sigma, A)$ *for every cycle* σ *in* D_A.

Theorem 4.3 *Let* $A \in \overline{\mathbb{R}}^{n \times n}$ *and* $\alpha \in \mathbb{R}$. *Then* $\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A)$ *for any* $\alpha \in \mathbb{R}$. *Hence* $(\lambda(A))^{-1} \otimes A$ *is definite whenever* $\lambda(A) > \varepsilon$.

The matrix $(\lambda(A))^{-1} \otimes A$ will be denoted by A_λ.

Computation of the maximum cycle mean from the definition is difficult except for small matrices since the number of cycles is prohibitively large in general. The task of finding the maximum cycle mean of a matrix was studied also in combinatorial optimisation quite independently of max-algebra. Publications presenting a method are e.g. [50], [25], [30], [45], [44]. One of the first and simplest was Vorobyov's formula

$$\lambda(A) = \max_{k \in N} \max_{i \in N} \frac{a_{ii}^{[k]}}{k}$$

where $A^k = \left(a_{ij}^{[k]} \right), k \in N$.

Example 4.2 *For the matrix A of Example 4.1 we get*

$$A^2 = \begin{pmatrix} 4 & 1 & 4 \\ 8 & 5 & 4 \\ 6 & 6 & 5 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 9 & 6 & 5 \\ 9 & 9 & 8 \\ 10 & 7 & 9 \end{pmatrix},$$

hence $\lambda(A) = \max(1, 5/2, 9/3) = 3$.

Perhaps the best known method today is *Karp's algorithm* [44] based on Theorem 4.4 below which finds the maximum cycle mean of an $n \times n$ matrix A in $O(n|E|)$ time where E is the set of arcs of D_A. Note that for the computation of the maximum cycle mean of a matrix we may assume without loss of generality that A is irreducible since any cycle is wholly contained in one strongly connected component and, as already mentioned, all strongly connected components can be recognised in linear time [49]. Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ and $s \in N$ be an arbitrary fixed node of $D_A = (N, E, (a_{ij}))$. For every $j \in N$, and every positive integer k we define $F_k(j)$ as the maximum weight of an $s - j$ path of length k; if no such path exists then $F_k(j) = \varepsilon$.

Theorem 4.4 *(Karp) If* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ *is irreducible then*

$$\lambda(A) = \max_{j \in N} \min_{k \in N} \frac{F_{n+1}(j) - F_k(j)}{n + 1 - k}. \tag{5}$$

Note that there are also other, fast methods for finding the maximum cycle mean for general matrices whose performance bound is not known. See for instance Howard's algorithm or the power method [2], [35], [40], [8], [32], [33].

4.2 The transitive closures

Given $A \in \overline{\mathbb{R}}^{n \times n}$ we define the infinite series

$$\Gamma(A) = A \oplus A^2 \oplus A^3 \oplus \dots \tag{6}$$

and

$$\Delta(A) = I \oplus \Gamma(A) = I \oplus A \oplus A^2 \oplus A^3 \oplus \dots \tag{7}$$

If these series converge, the matrix $\Gamma(A)$ is called the *weak transitive closure* of A, $\Delta(A)$ is the *strong transitive closure* of A. These names are motivated by the digraph representation if A is a $\{0, -1\}$ matrix since it is readily seen that the existence of arcs (i, j) and (j, k) in $Z_{\Gamma(A)}$ implies that also the arc (i, k) exists.

The matrices $\Gamma(A)$ and $\Delta(A)$ are of fundamental importance in max-algebra. This follows from the fact that they enable us to efficiently describe ALL non-trivial solutions (if any) to

$$A \otimes x = x \tag{8}$$

in the case of $\Gamma(A)$ and ALL finite solutions to

$$A \otimes x \leq x \qquad (9)$$

in the case of $\Delta(A)$. As a consequence (see Chapter 6) the weak transitive closure enables us to find all eigenvectors of a matrix and the strong transitive closure all finite solutions (called *subeigenvectors*) to

$$A \otimes x \leq \lambda \otimes x$$

where $\lambda \in \overline{\mathbb{R}}$ (see Theorem 4.5 below). The possibility of finding all solutions is an important feature of max-algebra and it provides a powerful mechanism for solving a range of problems whose solution would otherwise be awkward [14].

We will first show how $\Gamma(A)$ and $\Delta(A)$ can be used for finding one solution to (8) and (9), respectively. Then we describe all finite solutions to (9) using $\Delta(A)$. The description of all solutions to (8) will follow from the theory presented in Chapter 6.

Consider the matrix $A^2 = A \otimes A$: its elements are

$$\sum_{k \in N}^{\oplus} a_{ik} \otimes a_{kj} = \max_{k \in N}(a_{ik} + a_{kj}),$$

that is the weights of the heaviest $i - j$ paths of length 2 (if any) for all $i, j \in N$. Similarly the elements of A^k $(k = 1, 2, ...)$ are the weights of heaviest paths of length k for all pairs of nodes. Therefore the matrix $\Gamma(A)$ (if the infinite series converges) represents the weights of heaviest paths of any length for all pairs of nodes and motivated by this fact it is usually called the *metric matrix* corresponding to the matrix A [25]. Note that $\Delta(A)$ is often called the *Kleene star* [42].

If $\lambda(A) \leq 0$ then all cycles in D_A have non-positive weights and so by removing from a path a subpath which is a cycle will result in a path of a weight not less than that of the original path. Since every $i - j$ path of length greater than n contains a subpath which is a cycle, after a finite number of cycle deletions we can find an $i - j$ path of length n or less, whose weight is not less than that of the original path. Hence we have:

$$A^k \leq A \oplus A^2 \oplus ... \oplus A^n \quad \text{for every } k \geq 1 \qquad (10)$$

and therefore $\Gamma(A)$ for any matrix with $\lambda(A) \leq 0$, and in particular for definite matrices, exists and is equal to $A \oplus A^2 \oplus ... \oplus A^n$. On the other hand if $\lambda(A) > 0$ then a positive cycle in D_A exists and (6) diverges. We have proved:

Proposition 4.1 *Let $A \in \overline{\mathbb{R}}^{n \times n}$. Then (6) converges if and only if $\lambda(A) \leq 0$. If $\lambda(A) \leq 0$ then*

$$\Gamma(A) = A \oplus A^2 \oplus ... \oplus A^k$$

for every $k \geq n$. If A is also irreducible and $n > 1$ then $\Gamma(A)$ is finite.

A matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is called *increasing* if $a_{ii} \geq 0$ for all $i \in N$. Obviously, $A = I \oplus A$ when A is increasing and so then there is no difference between $\Gamma(A)$ and $\Delta(A)$.

Lemma 4.3 *If $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is increasing then $x \leq A \otimes x$ for every $x \in \overline{\mathbb{R}}^n$. Hence*

$$A \leq A^2 \leq A^3 \leq .... \tag{11}$$

A matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ is called *strongly definite* if it is definite and increasing. Since the diagonal entries of A are the weights of cycles (loops) we have that $a_{ii} = 0$ for all $i \in N$ if A is strongly definite. From (10) and (11) we deduce:

Proposition 4.2 *If $A \in \overline{\mathbb{R}}^{n \times n}$ is strongly definite then*

$$\Gamma(A) = A^{n-1} = A^n = A^{n+1} = ...$$

The matrix $\Delta(A)$ also has some remarkable properties. A key to understanding these is Lemma 1.1 which immediately implies another formula:

$$\Delta(A) = \Gamma(I \oplus A). \tag{12}$$

Proposition 4.3 *If $A \in \overline{\mathbb{R}}^{n \times n}$, $\lambda(A) \leq 0$ then*

$$\Delta(A) = I \oplus A \oplus ... \oplus A^{n-1}, \tag{13}$$

$$(\Delta(A))^k = \Delta(A) \tag{14}$$

for every $k \geq 1$ and

$$A \otimes \Delta(A) = \Gamma(A). \tag{15}$$

Theorem 4.5 *[24], [34], [47], [12] Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}, \lambda(A) > \varepsilon$. Then*

(a) *$A \otimes x \leq \lambda \otimes x$ has a finite solution if and only if $\lambda(A) \leq \lambda$.*

(b) *If $\lambda(A) \leq \lambda$ then*

$$A \otimes x \leq \lambda \otimes x, x \in \mathbb{R}^n$$

if and only if

$$x = \Delta(\lambda^{-1} \otimes A) \otimes u, u \in \mathbb{R}^n.$$

We finish this section with computational observations. The product of two $n \times n$ matrices from the definition uses $O(n^3)$ operations of $\oplus$ and $\otimes$ and unlike in conventional linear algebra a faster way of finding this product does not seem to be known. This implies that the computation of $\Gamma(A)$ for a matrix A with $\lambda(A) \leq 0$ from the definition needs $O(n^4)$ operations. However, the classical Floyd-Warshall algorithm for finding the all-pairs shortest/longest distances can do this job in $O(n^3)$ time.

Example 4.3 *For the matrix A of Example 4.1 we have $\lambda(A) = 3$, hence by subtracting 3 from every entry of A we obtain the definite matrix A_λ :*

$$\begin{pmatrix} -5 & -2 & -6 \\ 0 & -3 & 0 \\ 2 & -1 & -2 \end{pmatrix}.$$

We may calculate $\Gamma(A_\lambda)$ from the definition as $A_\lambda \oplus A_\lambda^2 \oplus A_\lambda^3$. Since

$$A_\lambda^2 = \begin{pmatrix} -2 & -5 & -2 \\ 2 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}, A_\lambda^3 = \begin{pmatrix} 0 & -3 & -4 \\ 0 & 0 & -1 \\ 1 & -2 & 0 \end{pmatrix}$$

we see that

$$\Gamma(A_\lambda) = \begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Alternatively we may use Algorithm Floyd-Warshall:

$$A_\lambda = \begin{pmatrix} -5 & -2 & -6 \\ 0 & -3 & 0 \\ 2 & -1 & -2 \end{pmatrix} \xrightarrow{p=1} \begin{pmatrix} -5 & -2 & -6 \\ 0 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow{p=2} \begin{pmatrix} -2 & -2 & -2 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \xrightarrow{p=3} \begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

4.3   Dual operators and conjugation

Other tools that help to overcome the difficulties caused by the absence of subtraction and matrix inversion are the *dual pair of operations* $(\oplus', \otimes')$ and the *matrix conjugation* respectively [24], [25]. These are defined as follows. For $a, b \in \overline{\overline{\mathbb{R}}}$ set

$$a \oplus' b = \min(a, b),$$

$$a \otimes' b = a + b \ \text{ if } \{a, b\} \neq \{-\infty, +\infty\}$$

and

$$(-\infty) \otimes' (+\infty) = +\infty = (+\infty) \otimes' (-\infty).$$

The pair of operations $(\oplus', \otimes')$ is extended to matrices (including vectors)

in the same way as $(\oplus, \otimes)$ and it is easily verified that all properties described in Section 1 hold dually if $\oplus$ is replaced by $\oplus'$ and $\otimes$ by $\otimes'$.

The *conjugate* a^* of $a \in \overline{\overline{\mathbb{R}}}$ is defined as follows: If $a \in \mathbb{R}$ then $a^* = a^{-1}$ and $\infty^* = -\infty, (-\infty)^* = \infty$. The *conjugate* of $A = (a_{ij}) \in \overline{\overline{\mathbb{R}}}^{m \times n}$ is $A^* = (b_{ij}) \in$

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$\overline{\overline{\mathbb{R}}}^{n \times m}$, where $b_{ij} = (a_{ji})^*$ for every $i \in M, j \in N$. We will denote b_{ij} by a_{ij}^*. Hence $A^* = -A^T$ if $A \in \mathbb{R}^{m \times n}$. The significance of conjugation is indicated by the following.

Theorem 4.6 *[24] If $A \in \overline{\overline{\mathbb{R}}}^{m \times n}, b \in \overline{\overline{\mathbb{R}}}^m$ and $x \in \overline{\overline{\mathbb{R}}}^n$ then*

$$A \otimes x \leq b \text{ if and only if } x \leq A^* \otimes' b.$$

Corollary 4.1 *If $A \in \overline{\overline{\mathbb{R}}}^{m \times n}$, $b \in \overline{\overline{\mathbb{R}}}^m$ and $c \in \overline{\overline{\mathbb{R}}}^n$ then*

(a) *$A^* \otimes' b$ is the greatest solution to $A \otimes x \leq b$, that is*

$$A \otimes (A^* \otimes' b) \leq b$$

(b) *$A \otimes x = b$ has a solution if and only if $\overline{x}$ is a solution and*

(c)
$$A \otimes (A^* \otimes' (A \otimes c)) = A \otimes c.$$

The vector $\overline{x} = A^* \otimes' b$ will be called the *principal solution* to $A \otimes x \leq b$ and $A \otimes x = b$.

5   Subspaces, generators, extremals and bases

Here we provide a brief overview of the theory of max-linear subspaces, independence and bases. This presentation follows the lines of [13]. The results of this section have been proved in [25], [41], [43] and [52].

Let $S \subseteq \overline{\mathbb{R}}^n$. The set S is called a *max-algebraic subspace* if $\alpha \otimes u \oplus \beta \otimes v \in S$ for every $u, v \in S$ and $\alpha, \beta \in \overline{\mathbb{R}}$. The adjective "max-algebraic" will usually be omitted.

A vector v is called a *max combination* of S if

$$v = \sum\nolimits_{x \in S}^{\oplus} \alpha_x \otimes x, \alpha_x \in \overline{\mathbb{R}} \tag{16}$$

where only a finite number of α_x are finite. The set of all max combinations of S is denoted by $span\,(S)$. We set $span\,(\emptyset) = \{\varepsilon\}$. It is easily seen that $span\,(S)$ is a subspace. If $span\,(S) = T$ then S is called a *set of generators* for T.

A vector $v \in S$ is called an *extremal in S* if $v = u \oplus w$ for $u, v \in S$ implies $v = u$ or $v = w$. Clearly, if $v \in S$ is an extremal in S and $\alpha \in \mathbb{R}$ then $\alpha \otimes v$ is also an extremal in S.

Let $v = (v_1, ..., v_n)^T \in \overline{\mathbb{R}}^n, v \neq \varepsilon$. The *max-norm* or just *norm* of v is $\|v\| = \max(v_1, ..., v_n)$; v is called *scaled* if $\|v\| = 0$. The set S is called *scaled* if all its elements are scaled.

The set S is called *dependent* if v is a max combination of $S - \{v\}$ for some $v \in S$. Otherwise S is *independent*. The set S is called *totally dependent* if every

$v \in S$ is a max combination of $S - \{v\}$. Note that $\emptyset$ is both independent and totally dependent and $\{\varepsilon\}$ is totally dependent.

Let $S, T \subseteq \overline{\mathbb{R}}^n$. The set S is called a *basis* of T if it is an independent set of generators for T. The set $\{e^i; i = 1, ..., n\}$ defined by

$$e^i_j = \left\{ \begin{array}{ll} 0 & \text{if } j = i \\ \varepsilon & \text{if } j \neq i \end{array} \right.$$

is a basis of $\overline{\mathbb{R}}^n$; it will be called *standard*.

Theorem 5.1 *Let E be the set of scaled extremals in a subspace T. Let $S \subseteq T$ consist of scaled vectors. Then the following are equivalent:*

(a) *S is a minimal set of generators for T.*

(b) *$S = E$ and S generates T.*

(c) *S is a basis for T.*

Theorem 5.1 shows that if a subspace has a (scaled) basis then it must be its set of (scaled) extremals, hence the basis is essentially unique. Note that a maximal independent set in a subspace T may not be a basis for T as is shown by the following example.

Example 5.1 *Let $T \subseteq \overline{\mathbb{R}}^2$ consist of all $(x_1, x_2)^T$ with $x_1 \geq x_2 > \varepsilon$. If $0 > a > b > \varepsilon$ then $\{(0, a)^T, (0, b)^T\}$ is a maximal independent set in T but it does not generate T.*

We now deduce a few corollaries of Theorem 5.1. The first one can be found in [31], [43] and [48].

Corollary 5.1 *If T is a finitely generated subspace then its set of scaled extremals is the unique scaled basis for T.*

Note that terminology varies in the max-algebraic literature and, for instance, extremals are called 'vertices' in [31], [43] and 'irreducible elements' in [51].

Next corollaries are related to totally dependent sets.

Corollary 5.2 *If S is a non-empty scaled totally dependent set then S is infinite.*

Corollary 5.3 *Let $T \subseteq \overline{\mathbb{R}}^n$ be a subspace. Then the following are equivalent:*

1. *There is no extremal in T.*

2. *There exists a totally dependent set of generators for T.*

3. *Every set of generators for T is totally dependent.*

A subspace S in $\overline{\mathbb{R}}^n$ is called *open* if $S - \{\varepsilon\}$ is open in the Euclidean topology.

Corollary 5.4 *Let $T \subseteq \overline{\mathbb{R}}^n$ be a subspace. If $T - \{\varepsilon\}$ is open and does not contain any vector of the standard basis then every generating set for T is totally dependent (and hence T has no basis).*

An example of an open subspace is $\mathbb{R}^n \cup \{\varepsilon\}$. For this particular case Corollary 5.4 was proved in [27]. Another example consists of all vectors $(a, b)^T$ with $a, b \in \mathbb{R}, a > b$.

More geometric and topological properties can be found in [13], [37], [20], [21], [19] and [41].

We have seen a number of corollaries of the key result, Theorem 5.1. We shall now link the first of these corollaries, Corollary 5.1 to column spaces of matrices. As usual the *column space* of a matrix $A \in \overline{\mathbb{R}}^{m \times n}$ is the subspace

$$Col(A) = \left\{ \sum_{j \in N}^{\oplus} x_j \otimes A_j; x_j \in \overline{\mathbb{R}} \right\} = \left\{ A \otimes x; x \in \overline{\mathbb{R}}^n \right\}$$

Recall that by finding a solution to a system $A \otimes x = b$ we prove that a vector (b) is in the subspace generated by the columns of A. An obvious task then is to find a basis of this subspace. Corollary 5.1 guarantees that such a basis exists and is unique up to scalar multiples of its elements. Note that for a formal proof we would have to first remove repeated columns as they would be indistinguishable in a set of columns, but they may be re-instated after deducing the uniqueness of the basis since the expression "multiples of a vector v" also covers vectors identical with v. We summarise:

Theorem 5.2 *For every $A \in \overline{\mathbb{R}}^{m \times n}$ there is a matrix $B \in \overline{\mathbb{R}}^{m \times k}, k \leq n$, consisting of some columns of A such that no two columns of B are equal and the set of column vectors of B is a basis of $Col(A)$. This matrix B is unique up to the order and scalar multiples of its columns.*

It remains to show how to find a basis of the column space of a matrix, say A. If a column, say A_k is a max combination of the remaining columns and A' arises from A by removing A_k then $Col(A) = Col(A')$ since in every max combination of the columns of A, the vector A_k may be replaced by a max combination of the other columns, that is columns of A'. By repeating this process until no vector can be found that would be a max combination of the remaining columns, we arrive at a set that satisfies both requirements in the definition of a basis. Every check of independence is equivalent to solving an $m \times (n-1)$ one sided system and can therefore be performed using $O(mn)$ operations, thus the whole process is $O(mn^2)$. Although asymptotically equally efficient, a method called the $\mathcal{A}$-test, essentially described in the following theorem, is more compact:

Theorem 5.3 *[25] Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{m \times n}$ and $\mathcal{A}$ be the matrix arising from $A^* \otimes' A$ after replacing the diagonal entries by ε. Then for all $j \in N$ the vector A_j is equal to the jth column of $A \otimes \mathcal{A}$ if and only if A_j is a max combination of the other columns of A. The elements of the jth column of $\mathcal{A}$ then provide the coefficients to express the max combination.*

Example 5.2 *Let*

$$A = \begin{pmatrix} 1 & 1 & 2 & \varepsilon & 5 \\ 1 & 0 & 4 & 1 & 5 \\ 1 & \varepsilon & -1 & 1 & 0 \end{pmatrix}.$$

Then

$$A^* \otimes' A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 0 & -\varepsilon \\ -2 & -4 & 1 \\ -\varepsilon & -1 & -1 \\ -5 & -5 & 0 \end{pmatrix} \otimes' \begin{pmatrix} 1 & 1 & 2 & \varepsilon & 5 \\ 1 & 0 & 4 & 1 & 5 \\ 1 & \varepsilon & -1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & -2 & 0 & -1 \\ 0 & \varepsilon & 1 & \varepsilon & 4 \\ -3 & \varepsilon & 0 & \varepsilon & 1 \\ 0 & \varepsilon & -2 & \varepsilon & -1 \\ -4 & \varepsilon & -3 & \varepsilon & 0 \end{pmatrix}.$$

Hence

$$A \otimes \mathcal{A} = \begin{pmatrix} 1 & 0 & 2 & 1 & 5 \\ 1 & \cdot & 2 & \cdot & 5 \\ 1 & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

We deduce

$$\begin{aligned} A_1 &= 0 \otimes A_2 \oplus -3 \otimes A_3 \oplus 0 \otimes A_4 \oplus -4 \otimes A_5 \\ A_5 &= -1 \otimes A_1 \oplus 4 \otimes A_2 \oplus 1 \otimes A_3 \oplus -1 \otimes A_4 \end{aligned}$$

and the basis of $Col(A)$ *is* $\{A_2, A_3, A_4\}$.

The number of vectors in a basis (and therefore in every basis) of a finitely generated subspace T is called the *dimension* of T, notation $\dim(T)$. Unlike in linear algebra, the dimensions of max-algebraic subspaces are unrelated to the numbers of components of the vectors in these subspaces. This has been observed long ago and the following two statements describe the anomaly.

Theorem 5.4 *[25] Let $m \geq 3$ and $k \geq 2$. There exist k vectors in $\overline{\mathbb{R}}^m$ none of which is a max combination of the others.*

Theorem 5.5 *[25] Every real $2 \times n$ matrix has two columns such that all other columns are a max combination of these two columns.*

6   The eigenproblem

Given $A \in \overline{\mathbb{R}}^{n \times n}$, the task of finding the vectors $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ (*eigenvectors*) and scalars $\lambda \in \overline{\mathbb{R}}$ (*eigenvalues*) satisfying

$$A \otimes x = \lambda \otimes x \qquad (17)$$

is called the (max-algebraic) *eigenproblem*. For some applications it may be sufficient to find one eigenvalue-eigenvector pair, however we show how to efficiently find all eigenvalues and eigenvectors for any matrix A.

The eigenproblem is of key importance in max-algebra. It has been studied since the 1960's [23] in connection with the analysis of the steady-state behaviour of production systems. Full solution of the eigenproblem in the case of irreducible matrices has been presented in [25] and [38], see also [50]. A general spectral theorem for reducible matrices has appeared in [35] and [3], and partly in [17].

Unless stated otherwise, we assume everywhere that $n \geq 1$ is an integer, $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ and $\lambda \in \overline{\mathbb{R}}$. Let us define

$$
\begin{aligned}
V(A, \lambda) &= \{x \in \overline{\mathbb{R}}^n; A \otimes x = \lambda \otimes x\}, \\
\Lambda(A) &= \{\lambda \in \overline{\mathbb{R}}; V(A, \lambda) \neq \{\varepsilon\}\}, \\
V(A) &= \bigcup_{\lambda \in \Lambda(A)} V(A, \lambda), \\
V^+(A, \lambda) &= V(A, \lambda) \cap \mathbb{R}^n, \\
V^+(A) &= V(A) \cap \mathbb{R}^n.
\end{aligned}
$$

Note that if $A = \varepsilon$ then $\Lambda(A) = \{\varepsilon\}$ and $V(A) = \overline{\mathbb{R}}^n$.

We also denote $E(A) = \{i \in N; \exists \sigma = (i = i_1, ..., i_k, i_1) : \mu(\sigma, A) = \lambda(A)\}$. The elements of $E(A)$ are called *critical nodes*. A cycle σ is called *critical* if $\mu(\sigma, A) = \lambda(A)$. *The critical digraph* of A is the digraph $C(A)$ with the set of nodes N; the set of arcs is the union of the sets of arcs of all critical cycles. It is well known that all cycles in a critical digraph are critical [2]. Two nodes i and j in $C(A)$ are called *equivalent* (notation $i \sim j$) if i and j belong to the same critical cycle of A. Clearly, $\sim$ constitutes a relation of equivalence in N.

6.1 The principal eigenvalue and corresponding eigenvectors

Note that if $\lambda(A) = \varepsilon$ then $\Lambda(A) = \{\varepsilon\}$ and the eigenvectors of A are exactly the vectors $(x_1, ..., x_n)^T \in \overline{\mathbb{R}}^n$ such that $x_j = \varepsilon$ whenever the jth column of A is not ε (clearly in this case at least one column of A is ε). We will therefore usually assume that $\lambda(A) > \varepsilon$.

Theorem 6.1 *[25][1]* $\lambda(A)$ *is an eigenvalue for any matrix* $A \in \overline{\mathbb{R}}^{n \times n}$. *If* $\lambda(A) > \varepsilon$ *then up to n eigenvectors corresponding to* $\lambda(A)$ *can be found among the columns of* $\Gamma(A_\lambda)$. *More precisely every column of* $\Gamma(A_\lambda)$ *with zero diagonal entry is an eigenvector of A with corresponding eigenvalue* $\lambda(A)$ *and we obtain a basis of* $V(A, \lambda(A))$ *by taking exactly one g_k for each equivalence class in* $(E(A), \sim)$.

It follows that $\lambda(A)$ is of a special significance as an eigenvalue: It is an eigenvalue for every matrix. We will also show in the next section that it is the only eigenvalue whose corresponding eigenvectors may be finite (Theorem 6.2) and it will follow from the Spectral Theorem (Theorem 6.7 below) that $\lambda(A)$ is the greatest eigenvalue. We will therefore call $\lambda(A)$ the *principal eigenvalue of* A and the subspace $V(A, \lambda(A))$ will be called the *principal eigenspace* of A. The dimension of this subspace will be denoted $\dim(A)$. Note that $\dim(A)$ can be found in $O(n^3)$ time.

Example 6.1 *Consider the matrix*

$$
A = \begin{pmatrix}
7 & 9 & 5 & 5 & 3 & 7 \\
7 & 5 & 2 & 7 & 0 & 4 \\
8 & 0 & 3 & 3 & 8 & 0 \\
7 & 2 & 5 & 7 & 9 & 5 \\
4 & 2 & 6 & 6 & 8 & 8 \\
3 & 0 & 5 & 7 & 1 & 2
\end{pmatrix}.
$$

The maximum cycle mean is 8 attained by three critical cycles: $(1, 2, 1)$, $(5, 5)$ and $(4, 5, 6, 4)$. Thus $\lambda(A) = 8$, $\dim(A) = 2$ and

$$
\Gamma(A_\lambda) = \begin{pmatrix}
0 & 1 & -1 & 0 & 1 & 1 \\
-1 & 0 & 2 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 1 \\
-1 & 0 & -1 & 0 & 1 & 1 \\
-2 & -1 & -2 & -1 & 0 & 0 \\
-2 & -1 & -2 & -1 & 0 & 0
\end{pmatrix}.
$$

$C(A)$ has two strongly connected components, one with the node set $\{1, 2\}$, the other one with the node set $\{4, 5, 6\}$. Hence first and second column of $\Gamma(A_\lambda)$ are multiples of each other and similarly the fourth, fifth and sixth columns. For the basis of $V(A) = V^+(A)$ we may take for instance the first and fourth column.

Example 6.2 *Consider the matrix*

$$
A = \begin{pmatrix}
0 & 3 & & \\
1 & -1 & & \\
& & 2 & \\
& & & 1
\end{pmatrix}
$$

where the missing entries are ε. Then $\lambda(A) = 2$, $E(A) = \{1, 2, 3\}$, $1 \sim 2$, $\dim(A) = 2$. We can compute

$$
\Gamma(A_\lambda) = \begin{pmatrix}
0 & 1 & & \\
-1 & 0 & & \\
& & 0 & \\
& & & -1
\end{pmatrix},
$$

hence a basis of the principal eigenspace is e.g.

$$\{g_2, g_3\} = \left\{ (1, 0, \varepsilon, \varepsilon)^T, (\varepsilon, \varepsilon, 0, \varepsilon)^T \right\}.$$

6.2 Finite eigenvectors

The following fundamental result shows that $\lambda(A)$ is the only possible eigenvalue corresponding to finite eigenvectors. Note that if $A = \varepsilon$ then every finite vector of a suitable dimension is an eigenvector of A and all correspond to the eigenvalue $\lambda(A) = \varepsilon$.

Theorem 6.2 *[25] Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$. If $A \neq \varepsilon$ and $V^+(A) \neq \emptyset$ then $\lambda(A) > \varepsilon$ and $A \otimes x = \lambda(A) \otimes x$ for every $x \in V^+(A)$.*

The next statement provides a simple criterion for the existence of a finite eigenvector.

Theorem 6.3 *Let $A \in \overline{\mathbb{R}}^{n \times n}$. If $\lambda(A) > \varepsilon$ and $\Gamma(A_\lambda) = (g_1, ..., g_n)$ then*

$$V^+(A) \neq \emptyset \iff \sum_{j \in E(A)}^{\oplus} g_j \in \mathbb{R}^n.$$

Since g_{ij} is the greatest weight of $i - j$ paths in D_{A_λ} we readily deduce a classical result:

Corollary 6.1 *(Cuninghame-Green [25]) Suppose $A \in \overline{\mathbb{R}}^{n \times n}, A \neq \varepsilon$. Then $V^+(A) \neq \emptyset$ if and only if the following are satisfied:*

1. $\lambda(A) > \varepsilon$.

2. *In D_A there is*
$$(\forall i \in N)(\exists j \in E(A)) i \to j.$$

Theorem 6.4 *Let $A \in \overline{\mathbb{R}}^{n \times n}$. If $\lambda(A) > \varepsilon, \Gamma(A_\lambda) = (g_1, ..., g_n)$ and $V^+(A) \neq \emptyset$ then*

$$V^+(A) = \left\{ \sum_{j \in E(A)}^{\oplus} \alpha_j \otimes g_j ; \alpha_j \in \mathbb{R} \right\}.$$

The set of finite eigenvectors can actually be generated in a slightly more efficient way (see Corollary 6.2), due to the following result.

Theorem 6.5 *[25] Suppose $A \in \overline{\mathbb{R}}^{n \times n}, \lambda(A) > \varepsilon$ and $\Gamma(A_\lambda) = (g_{ij}) = (g_1, ..., g_n)$. Then*

- $i \in E(A) \iff g_{ii} = 0$

- *If $i, j \in E(A)$ then $g_i = \alpha \otimes g_j$ for some $\alpha \in \mathbb{R}$ if and only if $i \sim j$.*

Corollary 6.2 *Let $A \in \overline{\mathbb{R}}^{n \times n}$. If $\lambda(A) > \varepsilon, \Gamma(A_\lambda) = (g_1, ..., g_n)$ and $V^+(A) \neq \emptyset$ then*

$$V^+(A) = \left\{ \sum_{j \in E^*(A)}^{\oplus} \alpha_j \otimes g_j; \alpha_j \in \mathbb{R} \right\} \tag{18}$$

where $E^*(A)$ is any maximal set of non-equivalent critical nodes of A. The size $|E^*(A)|$ is equal to the number of non-trivial strongly connected components of the critical digraph $C(A)$.

Remark 6.1 *Note that in (18) in general, g_j may or may not be in $V^+(A)$. Therefore the subspace $V^+(A)$ may or may not be finitely generated using vectors from $V^+(A)$ and hence there is no guarantee that $V^+(A)$ has a basis.*

Example 6.3 *Consider the matrix*

$$A = \begin{pmatrix} 0 & 3 & & \\ 1 & -1 & & \\ & & 2 & \\ & & 0 & 1 \end{pmatrix}$$

where the missing entries are ε. Then $\lambda(A) = 2$, $E(A) = \{1, 2, 3\}$, $\dim(A) = 2$. A finite eigenvector exists since an eigennode is accessible from every node (unlike in the slightly different Example 6.2). We can compute

$$\Gamma(A_\lambda) = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & -2 & -1 \end{pmatrix},$$

hence a basis of the principal eigenspace is $\left\{ (1, 0, \varepsilon, \varepsilon)^T, (\varepsilon, \varepsilon, 0, -2)^T \right\}$. All finite eigenvectors are a max combination of the vectors in the basis provided that both coefficients are finite. However, $V^+(A)$ has no basis.

The following classical complete solution of the eigenproblem for irreducible matrices can now be deduced:

Theorem 6.6 *(Cuninghame-Green [25]) Every irreducible matrix $A \in \overline{\mathbb{R}}^{n \times n}$ $(n > 1)$ has a unique eigenvalue equal to $\lambda(A)$ and*

$$V(A) - \{\varepsilon\} = V^+(A) = \left\{ \sum_{j \in E^*(A)}^{\oplus} \alpha_j \otimes g_j; \alpha_j \in \mathbb{R} \right\}$$

where $\Gamma(A_\lambda) = (g_1, ..., g_n)$ and $E^*(A)$ is any maximal set of non-equivalent critical nodes of A.

Remark 6.2 *Note that the 1×1 matrix $A = (\varepsilon)$ is irreducible and $V(A) - \{\varepsilon\} = V^+(A) = \mathbb{R}$.*

The fact that $\lambda(A)$ is the unique eigenvalue of an irreducible matrix A was proved in [23] and then independently in [50] for finite matrices. The description of $V^+(A)$ for irreducible matrices as given in Corollary 6.1 was also proved in [38].

Note that for an irreducible matrix A

$$V(A) = V^+(A) \cup \{\varepsilon\} = \{\Gamma(A_\lambda) \otimes z; z \in \overline{\mathbb{R}}^n, z_j = \varepsilon \text{ for all } j \notin E(A)\}.$$

Remark 6.3 *Since $\Gamma(A_\lambda)$ for an irreducible matrix A is finite, the generators of $V^+(A)$ are all finite if A is irreducible. Hence $V^+(A)$ has a basis in this case, which coincides with the basis of $V(A)$.*

Example 6.4 *Consider the irreducible matrix*

$$A = \begin{pmatrix} 0 & 3 & & 0 \\ 1 & -1 & 0 & \\ & 0 & 2 & \\ & & 0 & 1 \end{pmatrix}$$

where the missing entries are ε. Then $\lambda(A) = 2$, $E(A) = \{1, 2, 3\}$, $\dim(A) = 2$. We can compute

$$\Gamma(A_\lambda) = \begin{pmatrix} 0 & 1 & -4 & -2 \\ -1 & 0 & -5 & -3 \\ -3 & -2 & 0 & -5 \\ -5 & -4 & -2 & -1 \end{pmatrix},$$

hence a basis of the principal eigenspace is $\left\{ (1, 0, -2, -4)^T, (-4, -5, 0, -2)^T \right\}$.

6.3 Finding All Eigenvalues

First we introduce some notation that will be useful.

If

$$1 \le i_1 < i_2 < ... < i_k \le n, K = \{i_1, ..., i_k\} \subseteq N$$

then $A[K]$ denotes the principal submatrix

$$\begin{pmatrix} a_{i_1 i_1} & ... & a_{i_1 i_k} \\ ... & ... & ... \\ a_{i_k i_1} & ... & a_{i_k i_k} \end{pmatrix}$$

of the matrix $A = (a_{ij})$ and $x[K]$ denotes the subvector $(x_{i_1}, ..., x_{i_k})^T$ of the vector $x = (x_1, ..., x_n)^T \in \overline{\mathbb{R}}^n$.

If $D = (N, E)$ is a digraph and $K \subseteq N$ then $D[K]$ denotes the *induced subgraph* of D, that is

$$D[K] = (K, E \cap (K \times K)).$$

Obviously, $D_{A[K]} = D[K]$.

The symbol $A \sim B$ for matrices A and B means that A can be obtained from B by a simultaneous permutation of rows and columns. Clearly, in that case D_A can be obtained from D_B by a renumbering of the nodes. Hence if $A \sim B$ then A is irreducible if and only if B is irreducible.

Lemma 6.1 *If $A \sim B$ then $\Lambda(A) = \Lambda(B)$ and there is a bijection between $V(A)$ and $V(B)$.*

The following lemma gives a clear signal that also in max-algebra the Frobenius normal form will be useful for describing all eigenvalues.

Lemma 6.2 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}, \lambda \in \Lambda(A)$ and $x \in V(A, \lambda)$. If $x \notin V^+(A, \lambda)$ then $n > 1$,*

$$A \sim \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix},$$

$\lambda = \lambda(A^{(22)})$ and hence A is reducible.

Proposition 6.1 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$. Then $V(A) = V^+(A)$ if and only if A is irreducible.*

Every matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ can be transformed in linear time by simultaneous permutations of the rows and columns to a *Frobenius normal form* (FNF) [46]

$$\begin{pmatrix} A_{11} & \varepsilon & ... & \varepsilon \\ A_{21} & A_{22} & ... & \varepsilon \\ ... & ... & ... & ... \\ A_{r1} & A_{r2} & ... & A_{rr} \end{pmatrix} \tag{19}$$

where $A_{11}, ..., A_{rr}$ are irreducible square submatrices of A. If A is in an FNF then the corresponding partition of the node set N of D_A will be denoted as $N_1, ..., N_r$ and these sets will be called *classes (of A)*. It follows that each of the induced subgraphs $D_A[N_i]$ $(i = 1, ..., r)$ is strongly connected and an arc from N_i to N_j in D_A exists only if $i \geq j$. As a slight abuse of language we will also say for simplicity that $\lambda(A_{jj})$ is the eigenvalue of N_j.

If A is in an FNF, say (19), then the *reduced digraph*, notation C_A, is the digraph $(\{N_1, ..., N_r\}, \{(N_i, N_j); (\exists k \in N_i)(\exists \ell \in N_j)a_{k\ell} > \varepsilon\})$.

The symbol $N_i \to N_j$ means that there is a directed path from a node in N_i to a node in N_j in D_A (and therefore from each node in N_i to each node in N_j). Equivalently, there is a directed path from N_i to N_j in C_A.

If there are neither outgoing nor incoming arcs from or to an induced subgraph $C_A[\{N_{i_1}, ..., N_{i_s}\}]$ $(1 \leq i_1 < ... < i_s \leq r)$ and no proper subdigraph has this property then the submatrix

$$\begin{pmatrix} A_{i_1 i_1} & \varepsilon & ... & \varepsilon \\ A_{i_2 i_1} & A_{i_2 i_2} & ... & \varepsilon \\ ... & ... & ... & ... \\ A_{i_s i_1} & A_{i_s i_2} & ... & A_{i_s i_s} \end{pmatrix}$$
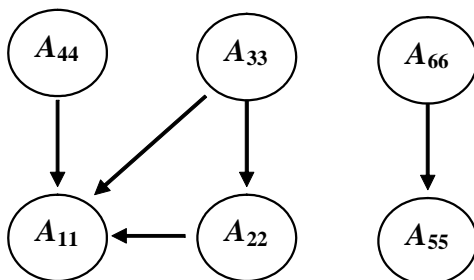
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Figure 1: Condensation digraph for matrix (4)

is called an *isolated superblock* (or just *superblock*). The induced subdigraph of C_A corresponding to an isolated superblock is a directed tree (although the underlying undirected graph is not necessarily acyclic). C_A is the union of a number of such directed trees. The nodes of C_A with no incoming arcs are called the *initial classes*, those with no outgoing arcs are called the *final classes*. Note that the directed tree corresponding to an isolated superblock may have several initial and final classes.

For instance the reduced digraph for the matrix

$$
\begin{pmatrix}
A_{11} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
* & A_{22} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
* & * & A_{33} & \varepsilon & \varepsilon & \varepsilon \\
* & \varepsilon & \varepsilon & A_{44} & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{55} & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & * & A_{66}
\end{pmatrix}
\tag{20}
$$

can be seen in Figure 1 (note that here and elsewhere the symbols $*$ indicate submatrices different from ε). It consists of two superblocks and six classes including three initial and two final ones.

Lemma 6.3 *If $x \in V(A), N_i \rightarrow N_j$ and $x[N_j] \neq \varepsilon$ then $x[N_i]$ is finite. In particular, $x[N_j]$ is finite.*

The following key result has appeared in the thesis [35] and [4]. The latter work refers to the report [3] for a proof.

Theorem 6.7 *(Spectral Theorem) Let (19) be an FNF of a matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$. Then*

$$
\Lambda(A) = \{\lambda(A_{jj}); \lambda(A_{jj}) = \max_{N_i \rightarrow N_j} \lambda(A_{ii})\}.
$$

Note that significant correlation exists between the max-algebraic spectral theory and that for non-negative matrices in linear algebra [47], [6], see also [46]. For instance the Frobenius normal form and accessibility between classes are

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essentially the same. The maximum cycle mean corresponds to the Perron root for irreducible (nonnegative) matrices and finite eigenvectors in max-algebra correspond to positive eigenvectors in the non-negative spectral theory. However there are also differences, see Remark 6.5 after Theorem 6.9 below.

Let A be in the FNF (19). If

$$\lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})$$

then A_{jj} (and also N_j or just j) will be called *spectral*. Thus $\lambda(A_{jj}) \in \Lambda(A)$ if j is spectral but not necessarily the other way round. The following corollaries of the spectral theorem are readily proved.

Corollary 6.3 *All initial classes of C_A are spectral.*

Corollary 6.4 $1 \leq |\Lambda(A)| \leq n$ *for every $A \in \overline{\mathbb{R}}^{n \times n}$.*

Corollary 6.5 $V(A) = V(A, \lambda(A))$ *if and only if all initial classes have the same eigenvalue $\lambda(A)$.*

Figure 2 shows a reduced digraph with 14 classes including two initial classes and four final ones. The numbers indicate the eigenvalues of the corresponding classes. Six bold classes are spectral, the others are not.

6.4   Finding All Eigenvectors

Note that the unique eigenvalue of every class (that is of a diagonal block of an FNF) can be found in $O(n^3)$ time by applying Karp's algorithm (see Section 1) to each block. The condition for identifying all spectral submatrices in an FNF provided in Theorem 6.7 enables us to find them in $O(r^2) \leq O(n^2)$ time by applying standard reachability algorithms to C_A.

Let $A \in \overline{\mathbb{R}}^{n \times n}$ be in the FNF (19), $N_1, ..., N_r$ be the classes of A and $R = \{1, ..., r\}$. Suppose $\lambda \in \Lambda(A), \lambda > \varepsilon$ and denote

$$I(\lambda) = \{i \in R; \lambda(N_i) = \lambda, N_i \text{ spectral}\}.$$

Similarly as in Section 1 we denote $\Gamma(\lambda^{-1} \otimes A) = (g_{ij}) = (g_1, ..., g_n)$. Note that $\Gamma(\lambda^{-1} \otimes A)$ may now include entries equal to $+\infty$. Let us denote

$$E(\lambda) = \bigcup_{i \in I(\lambda)} E(A_{ii}) = \{j \in N; g_{jj} = 0, j \in \bigcup_{i \in I(\lambda)} N_i\}.$$

Two nodes i and j in $E(\lambda)$ are called λ - *equivalent* (notation $i \sim_\lambda j$) if i and j belong to the same cycle of cycle mean λ.

Theorem 6.8 *Suppose $A \in \overline{\mathbb{R}}^{n \times n}$ and $\lambda \in \Lambda(A), \lambda > \varepsilon$. Then $g_j \in \overline{\mathbb{R}}^n$ for all $j \in E(\lambda)$ and a basis of $V(A, \lambda)$ can be obtained by taking one g_j for each $\sim_\lambda$ equivalence class.*
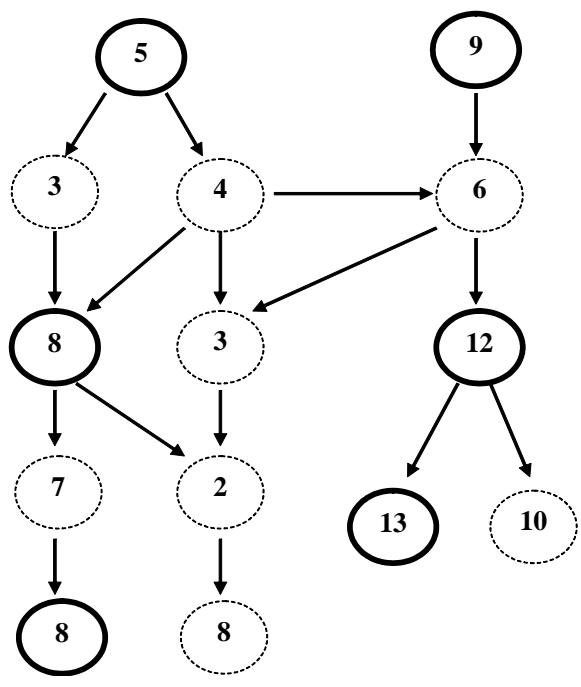
Figure 2: A condensation digraph with six spectral nodes

Corollary 6.6 *A basis of $V(A, \lambda)$ for $\lambda \in \Lambda(A)$ can be found using $O(n^3)$ operations and we have*

$$V(A, \lambda) = \{\Gamma(\lambda^{-1} \otimes A) \otimes z; z \in \overline{\mathbb{R}}^n, z_j = \varepsilon \text{ for all } j \notin E(\lambda)\}.$$

Theorem 6.9 *$V^+(A) \neq \emptyset$ if and only if $\lambda(A)$ is the eigenvalue of all final classes.*

Corollary 6.7 *$V^+(A) = \emptyset$ if and only if a final class has eigenvalue less than $\lambda(A)$.*

Remark 6.4 *Note that a final class with eigenvalue less than $\lambda(A)$ may not be spectral and so $\Lambda(A) = \{\lambda(A)\}$ is possible even if $V^+(A) = \emptyset$. For instance in the case of*

$$A = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}$$

we have $\lambda(A) = 1$, but $V^+(A) = \emptyset$.

Remark 6.5 *In the Perron-Frobenius theory of non-negative matrices, a class of a non-negative matrix is called basic if its spectral radius coincides with the spectral radius of the matrix. A classical result shows that a non-negative matrix has a positive eigenvector if and only if its basic and final classes coincide. In the max-algebraic setting we may define a class to be basic when its eigenvalue is $\lambda(A)$. Then, Theorem 6.9 shows that the existence of a finite eigenvector only requires all final classes to be basic; unlike in the Perron-Frobenius theory, there may also be non-final basic classes. For instance the non-negative matrix*

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

has two basic classes $\{1\}$ and $\{2\}$ and only one final class, namely $\{1\}$, thus it does not have a positive eigenvector. However, its max-algebraic counterpart

$$A = \begin{pmatrix} 0 & -\infty \\ 0 & 0 \end{pmatrix}$$

which satisfies the condition of Theorem 6.9 has a finite eigenvector (for instance $(0, 0)^T$). This fundamental discrepancy is due to the idempotency of $\oplus$ in max-algebra.

7 Some more max-algebra and open problems

7.1 Characteristic maxpolynomial

The *max-algebraic permanent* of A is defined as an analogue of the classical one:

$$maper(A) = \bigoplus_{\pi \in P_n} \bigotimes_{i \in N} a_{i,\pi(i)}$$

where P_n stands for the set of all permutations of the set N. In the conventional notation

$$maper(A) = \max_{\pi \in P_n} \sum_{i \in N} a_{i,\pi(i)}$$

which is the optimal value of the assignment problem for the matrix A. There are a number of efficient solution methods [10] for finding $maper(A)$, one of the best known is the Hungarian method of computational complexity $O(n^3)$. The set of all optimal permutations will be denoted by $ap(A)$, that is,

$$ap(A) = \{\pi \in P_n; maper(A) = \sum_{i \in N} a_{i,\pi(i)}\}.$$

The max-algebraic characteristic polynomial called *characteristic maxpolynomial* of $A \in \mathbb{R}^{n \times n}$ is defined [25] by

$$\chi_A(x) := maper(A \oplus x \otimes I).$$

Hence

$$\chi_A(x) = \delta_0 \oplus \delta_1 \otimes x \oplus \cdots \oplus \delta_{n-1} \otimes x^{(n-1)} \oplus x^{(n)}$$

for some $\delta_0, ..., \delta_{n-1} \in \mathbb{R}$ or, written using conventional notation

$$\chi_A(x) = \max\left(\delta_0, \delta_1 + x, \cdots, \delta_{n-1} + (n-1)x, nx\right).$$

Thus, viewed as a function of x, the characteristic maxpolynomial of a matrix A is a piecewise linear convex function whose slopes are from the set $\{0, 1, ..., n\}$. As a function, $\chi_A(x)$ can be found in $O(n^4)$ steps [9] but no efficient way seems to be known for finding all $\delta_0, \delta_1, \cdots, \delta_{n-1}$.

Open Problem P1: Given $A \in \mathbb{R}^{n \times n}$, is the problem of finding all $\delta_0, \delta_1, \cdots, \delta_{n-1}$ polynomially solvable or NP-complete?

It is known [28] that every max-algebraic polynomial can be factorized to linear factors in $O(n \log n)$ time. Thus the factors are of the form $x \oplus \beta$ where β is a real constant. The constants β are called the *corners* of the max-polynomial. The characteristic maxpolynomials have a remarkable property resembling the conventional characteristic polynomials:

Theorem 7.1 *[25] The greatest corner of $\chi_A(x)$ is $\lambda(A)$.*

Open Problem P2: What is the meaning of the other corners?

7.2 Alternative concepts of linear dependence/independence

$A = (a_{ij}) = (A_1, ..., A_n) \in \mathbb{R}^{m \times n}$ is said to have

- independent columns if none of the columns is a max combination of the remaining columns;

- Gondran-Minoux independent columns if

$$\sum_{j \in S}^{\oplus} \lambda_j \otimes A_j = \sum_{j \in T}^{\oplus} \lambda_j \otimes A_j$$

  does not holds for any real numbers λ_j and two non-empty, disjoint subsets S and T of N;

- strongly independent columns if $A \otimes x = b$ has a unique solution for at least one b.

Open Problem P3: Find efficient methods for checking Gondran-Minoux independence and for checking strong independence.

A square matrix is called regular (Gondran-Minoux regular, strongly regular) if its columns are independent (Gondran-Minoux independent, strongly independent).

Theorem 7.2 *[39], [11] Let A be a square matrix.*

(a) *A is strongly regular if and only if $ap(A)$ contains only one permutation.*

(b) *A is Gondran-Minoux regular if and only if all permutations in $ap(A)$ have the same parity.*

Corollary 7.1 *A regular $\implies$ A Gondran-Minoux regular $\implies$ A strongly regular.*

Corollary 7.2 *Given A, it is possible to check using $O(n^3)$ operations whether A is regular, Gondran-Minoux regular or strongly regular.*

7.3 Two-sided systems

Two-sided systems (TSS) of the form

$$A \otimes x \oplus b = C \otimes x \oplus d$$

can be easily transformed to homogenous systems

$$A \otimes x = B \otimes x$$

or to systems with separated variables

$$A \otimes x = B \otimes y.$$

The latter can be solved using the pseudopolynomial alternating method [26]. The problem is known to be in $NP \cap co - NP$ and hence the existence of a polynomial solution method is expected.

Conjecture 7.3 *Two-sided max-algebraic systems are solvable in polynomial time.*

The problem of finding λ and x satisfying the generalized eigenproblem

$$A \otimes x = \lambda \otimes B \otimes x$$

can be considered as a generalization of both the eigenproblem and TSS. This is one of the most challenging problems in max-algebra and no method seems to be available in general. A number of special cases can be solved [29]. If A, B are symmetric then there is at most one generalized eigenvalue [7].

7.4 Link to nonnegative matrices

Let A be an irreducible nonnegative matrix (in the ususual linear-algebraic sense) and let $\left\{A^k\right\}_{k=1}^{\infty}$ be the sequence of Hadamard powers of A. Let λ_k be the Perron root of A^k. It is known then that $\sqrt[k]{\lambda_k} \longrightarrow \lambda(A)$ in the max-times algebra [42]. Let x_k be the Perron vector of A^k. Since A may have several independent max-algebraic eigenvectors, it is not immediately clear to which one does the sequence $\{x_k\}_{k=1}^{\infty}$ converge.

Conjecture 7.4 $\{x_k\}_{k=1}^{\infty}$ *converges to the barycentre of the set of fundamental (max-algebraic) eigenvectors.*

7.5 Permutation problems

The following permutation problems in max-plus are NP-complete (when all entries are integer) [15]:

(PEV) Given a square matrix A and a vector x, is it possible to permute the components of x so that the arising vector is an eigenvector of A?

(PLS) Given a matrix A and a vector b, is it possible to permute the components of b so that for the arising vector b' the system $A \otimes x = b'$ is solvable?

We can of course formulate similar problems in conventional linear algebra and both are NP-complete, too. However, PEV for positive matrices is easily solvable since by Perron-Frobenius there is a unique positive eigenvector (up to a multiple). This gives rise to the following open problems:

Open Problem P4: Is PEV for non-negative matrices polynomially solvable or NP-complete?

Open Problem P5: Is PLS for positive/non-negative matrices polynomially solvable or NP-complete?

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